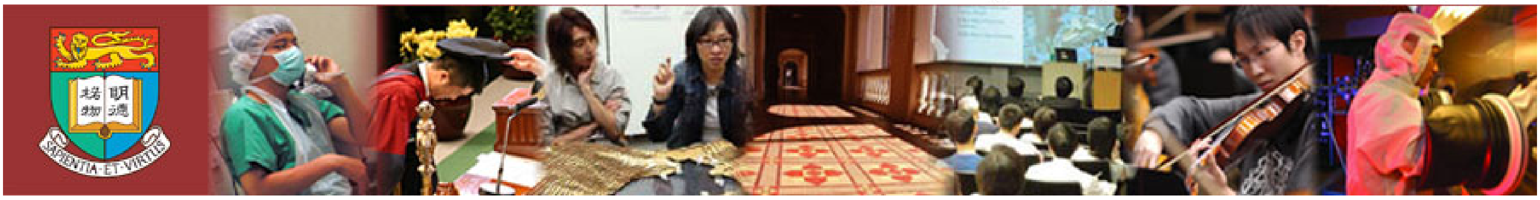

Optimal Doppler Frequency Estimators for Ultrasound and Optical Coherence Tomography

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***Abstract*—The Kasai autocorrelation estimator is widely used in Doppler optical coherence tomography and ultrasound to determine blood velocities. However, as a non-parametric estimator, it may not be optimal. Assuming an additive white Gaussian noise (AWGN) model, we show that the Kasai estimator variance is far from the Cramer-Rao lower bound. Moreover, paradoxically, the Kasai estimator performance degrades as the acquisition rate is increased. By contrast, the additive white Gaussian noise maximum likelihood estimator (AWGN MLE) variance asymptotically approaches the Cramer-Rao lower bound, making it a better estimator at high acquisition rates. Nevertheless, the Kasai estimator outperforms the AWGN MLE under moderate levels of multiplicative decorrelation noise, and could therefore be considered more robust. These findings motivate further work in maximum likelihood estimators under conditions of both additive and multiplicative noise.**

***Index Terms*—Cramer-Rao bounds, maximum likelihood estimation, frequency estimation, Doppler ultrasound, Doppler optical coherence tomography.**

I. Introduction

In this work we examine the statistical performance of the commonly used Kasai autocorrelation method [1], [2], which is used in Doppler ultrasound [3]–[5] and Doppler optical coherence tomography [6], [7]. Although it is fast and simple, with wide applicability due to its non-parametric nature, we show that it is not optimal under additive Gaussian noise conditions. We verify the result of Schmoll et al. [8] that increasing the acquisition rate decreases the Kasai estimator performance. In practice, the Kasai lag, or time lag at which the phase of the autocorrelation function is estimated, can be increased in order to lower the estimator variance [8]. This is clearly undesirable, as increasing the autocorrelation lag necessarily reduces the maximum measurable frequency by the Nyquist sampling theorem.

In comparison, we find that although the AWGN MLE is statistically optimal in the sense of asymptotic mean squared error (MSE) and variance [9], it is not as robust as the Kasai estimator. It is highly sensitive to outliers and deviations from noise model assumptions [10], [11] and its statistical optimality is only realized under a narrow range of experimental conditions. Actual OCT signals contain both decorrelation noise, or multiplicative noise [12], and additive shot noise. The multiplicative noise gives rise to spreading of the signal in the frequency domain and causes deviations from the assumption of additive noise. We find that, for moderate amounts of decorrelation noise, the Kasai estimator outperforms the AWGN MLE. Our work illustrates the advantages of maximum likelihood estimators, and underscores the sensitivity of such estimators to deviations from noise model assumptions.

II. Kasai Estimator

Kasai derived an estimator [1] to calculate Doppler shifts of continuous wave ultrasound signals. While Kasai derived this estimator in the continuous case, it is often utilized in its discrete form. The phase of the lag one autocorrelation function acts as an estimate of the phase change during this time interval. By dividing by the time interval, one obtains an estimate for the Doppler frequency.

In the continuous regime, the Doppler frequency is proportional to the time derivative of the autocorrelation function at time zero. For actual data, the autocorrelation function needs to be estimated as

$$R(\Delta t) \approx \hat{R}(\Delta t) = \sum_{n=1}^{N-1} s_{n+1} s_n^*. \tag{1}$$

Here s_n is the signal acquired at the nth time instant. If its complex phase is given by $\phi(\Delta t) = \angle\left[R(\Delta t)\right]$, then one can estimate the magnitude of the gradient at time zero by dividing the phase subtended by the time elapsed, Δt. This is the same as estimating the rate of change in phase, $\dot{\phi}(0)$, at time zero,

$$\bar{\Omega}_{\mathrm{k}} = \dot{\phi}(0) \approx \frac{\phi(\Delta t)}{\Delta t} = \frac{\angle\left[R(\Delta t)\right]}{\Delta t}. \tag{2}$$

Therefore, from (1) and (2), the estimator is given by

$$\hat{\Omega}_{\mathrm{k}} = \frac{\angle\left\{\sum_{n=1}^{N-1} |s_{n+1}||s_n| \exp[j(\phi_{n+1} - \phi_n)]\right\}}{\Delta t}. \tag{3}$$

This is a two-step estimation process. First, one estimates the autocorrelation function. Second, from the value at unit lag, one estimates the phase velocity.

As no assumptions are made about the noise statistics, the Kasai method provides reasonable estimates even in the presence of decorrelation noise [12]. It therefore has a wider applicability than a parametric method such as the maximum likelihood estimator. It is also computationally efficient and can therefore be implemented in real-time with very simple electronics. However its non-parametric nature means that *a priori* knowledge of the noise statistics is not utilized, resulting in sub-optimality. Here we show using simulations, that the Kasai estimator does not achieve the Cramer-Rao lower bound (CRLB) under AWGN assumptions and realistic SNRs.

III. Maximum Likelihood Estimator

The maximum likelihood estimator (MLE) is consistent, asymptotically unbiased, and asymptotically efficient. For frequency estimation under additive white noise assumptions it is equal to the position of peak of the power spectral density.

A. AWGN Model

To derive the AWGN MLE we need to consider a stationary OCT beam focussed at a single location. The signal would be obtained from the time evolution of the complex reflectance of this region. The signal can be represented as the sum of a rotating phasor and complex Gaussian noise. As the MLE is a parametric estimator, provided that the acquired signal is well described by the noise model, it is asymptotically efficient and unbiased.

If s_n is a single data point at time instant n, we represent the Doppler OCT data for measuring flow velocity as

$$s_n = |R| \exp[j(n\Omega\Delta t + \phi_R)] + z_n. \tag{4}$$

Here, $|R|\exp(j\phi_R)$ is the unknown complex constant reflectance, and $j = \sqrt{-1}$. We wish to estimate the Doppler frequency, Ω, from the signal. The time between measurements is $\Delta t = T/N$, where T is the total acquisition time and N is the total number of samples. The additive noise is given by z_n, which is circularly symmetric complex Gaussian. Each of its real and imaginary components is Gaussian with variance σ^2. As the noise is white, the z_n are independent and identically distributed (i.i.d.) with variance $2\sigma^2$.

B. AWGN Maximum Likelihood Estimator – Peak of Spectrum

The AWGN MLE, $\hat{\Omega}_{\text{MLE}}$, is obtained by choosing the values of the Doppler frequency, Ω, and reflectance phase, ϕ_R, that maximizes the real part of the inverse DFT of the (complex conjugate of the) signal.

$$\hat{\theta}_{\text{MLE}} = \underset{\mathbf{\Omega},\phi_{\mathbf{R}}}{\arg\max} \left(\text{Re}\left\{ |R| \sum_{n=1}^{N} s_n^* \exp\left[j\left(\frac{n\Omega T}{N} + \phi_R \right) \right] \right\} \right). \tag{5}$$

As we are not interested in ϕ_R, this is equivalent to finding frequency corresponding to the peak of the power spectral density (PSD).

C. Advantages and Drawbacks

The advantages of the MLE include asymptotic efficiency, consistency and asymptotic unbiasedness. However these desirable asymptotic properties may not be achieved for small samples. As a parametric method, the MLE performance may deteriorate in the presence of outliers or deviations from model assumptions. One example of such a deviation from model assumptions is the presence of multiplicative decorrelation noise.

IV. Cramer-Rao Lower Bound

The Cramer-Rao lower bound determines the minimum variance of an estimator of a deterministic parameter. Using an additive white Gaussian noise model, one can derive that the AWGN CRLB is given by

$$\text{Var}(\hat{\Omega}) \geq \text{CRLB}(\hat{\Omega}) = \frac{12N\sigma^2}{(N^2-1)|R|^2T^2} \approx \frac{12\sigma^2}{N|R|^2T^2}. \tag{6}$$

The last expression is valid for large N.

Here the AWGN CRLB for the estimate of the Doppler frequency is inversely proportional to the total number of samples, N, inversely proportional to the SNR, $|R|^2/2\sigma^2$, and inversely proportional to the square of the total acquisition time T. Further insight can be achieved by including the fact that, assuming a constant rate of detected photons (power), the shot-noise limited SNR is proportional to $\Delta t = T/N$. Under these conditions,

$$\text{CRLB}(\hat{\Omega}) \sim 1/T^3. \tag{7}$$

Thus, if the number of samples is sufficiently large, the SNR is shot noise limited, and the rate of detected photons (power) is constant, the CRLB is inversely proportional to the cube of the total acquisition time. More importantly, for large N, the CRLB becomes independent of N and also becomes independent of sampling rate. This behavior contrasts with the Kasai estimator, whose variance increases with increasing sampling rate.

V. Simulations

We ran simulations to estimate the variances and biases of the estimators. We generated 2000 instances of the data. The analog frequency was assumed to be $\Omega = 6\pi \times 10^3$ rad.s^{-1} for all simulations, corresponding to typical OCT Doppler shifts. The estimator variance and bias are not expected to be sensitive to analog frequency. The variance and bias of each estimator were estimated from this data. We define the SNR to be $|R|^2/2\sigma^2$.

A. Varying SNR

Fig. 1a shows that for a data length of $N = 32$, the AWGN MLE achieves the CRLB at an SNR of roughly -1 dB. The Kasai estimator slowly approaches the CRLB but is worse than the CRLB by more than 8 dB. In our simulations, except for those involving multiplicative noise, the biases of the estimators are too small to be significant, hence the values of the estimator variance and mean squared error (MSE) are practically the same.

B. Varying Acquisition Time and Acquisition Rate

We show in [2] that the MSE of the commonly-used Kasai estimator increases with increasing acquisition rate. This is true whether the SNR is kept constant as shown in Fig. 2, or if the detected photon rate (power) is kept constant, as shown in Fig. 3. This is non-intuitive and paradoxical behavior, since one would expect that increasing the sampling rate, while keeping the total number of detected photons the same, would

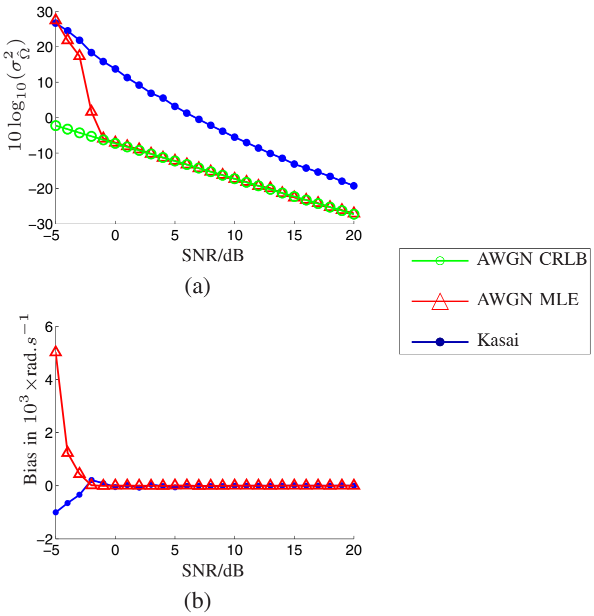

Fig. 1: (a) The sample variance of estimates in the presence of AWGN compared with the Cramer-Rao Lower Bound (CRLB), for a data length of $N = 32$, and an acquisition time of $T = 0.001s$. The AWGN MLE achieves the CRLB except for at low SNRs. The Kasai estimator is more than 8 dB worse than the AWGN MLE, for moderate to high SNRs. (b) Estimator bias in 10^3 radians per second with SNR. These values are too small to be significant, hence the variance of the estimators and the MSE can be considered to be the same.

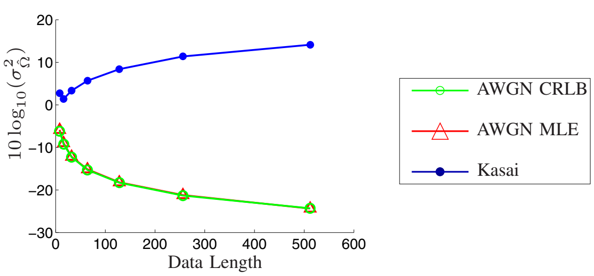

Fig. 2: MSE of Kasai and AWGN ML estimators against data length for a constant acquisition time of $T = 0.001$ s at an SNR of 5 dB. In practice, maintaining a constant SNR while increasing the acquisition rate would require increasing the detected photon rate (power). The Kasai estimator performance becomes worse with increasing acquisition rate (data length). The AWGN MLE performance improves with acquisition rate and achieves the CRLB for all data lengths above 32.

provide more information about parameters to be estimated. Sometimes the Kasai lag can be increased to achieve a more precise Kasai estimate; however this has the undesirable consequence of decreasing the maximum measurable Doppler frequency.

In contrast, we have shown that the CRLB for estimator variance, for large samples, is proportional to $1/T^3$, as shown in (7), and is independent of the sampling rate. Since the AWGN MLE variance approaches the CRLB asymptotically, it also asymptotically possesses this desirable property. For sufficiently large samples, the AWGN MLE performance is independent of the number of samples. Therefore, we conclude that the AWGN MLE is the estimator of choice for high speed Doppler OCT provided that additive noise dominates and multiplicative noise is negligible. One can reduce multiplicative noise by minimizing sources of decorrelation such as beam scanning.

C. Performance with Multiplicative Noise

To model multiplicative decorrelation noise we modified the signal from (4) to include a multiplicative term x_n,

$$s_n = [mx_n + 1]|R|\exp[j(n\Omega\Delta t + \phi_R)] + z_n, \qquad (8)$$

where x_n is a correlated complex Gaussian random variable. In our simulations we assume that additive noise is negligible and so we set $z_n = 0$. We predetermined the auto-covariance matrix of x_n to be real Toeplitz, with the first row (the auto-covariance function) being the values of a Gaussian profile from the non-negative domain. Its $1/e$ full-width is determined by the coherence time of the signal, and was set to be $4\Delta t$ for these simulations. The continuous parameter m is used to vary the relative amount of multiplicative noise compared with the static reflectivity, or noiseless signal. Thus the quantity multiplying the rotating phasor has a Rician distribution.

Figs. 4a and 4b show that the AWGN MLE performance deteriorates more rapidly than the Kasai estimator with an increasing proportion of multiplicative noise, even in the absence of additive noise. This suggests the Kasai estimator's greater robustness against decorrelation noise. In the AWGN dominant regime, the AWGN MLE performs better, but only has tolerance for low levels of decorrelation noise. In the multiplicative decorrelation noise dominant regime, the Kasai estimator performs better. Thus, care should be taken to ensure that maximum likelihood estimators are derived using valid model assumptions.

VI. Conclusion

In this paper we have discussed and demonstrated the relative advantages of the Kasai autocorrelation estimator and AWGN MLE. We have shown that the AWGN MLE is statistically optimal under additive white Gaussian noise conditions, and has a slight tolerance for low levels of multiplicative decorrelation noise. The Kasai estimator also gives unbiased estimates, but is sub-optimal under AWGN. However, the Kasai estimator outperforms the AWGN MLE with significant amounts of multiplicative noise.

Our work demonstrates that the MLE is asymptotically efficient. However, our work also underscores the sensitivity of the MLE to deviations from assumptions about the noise statistics. Ideally, it is better to formulate a more general MLE that incorporates both additive noise and the effects of decorrelation with an appropriate statistical model. This will be further explained in our future work.

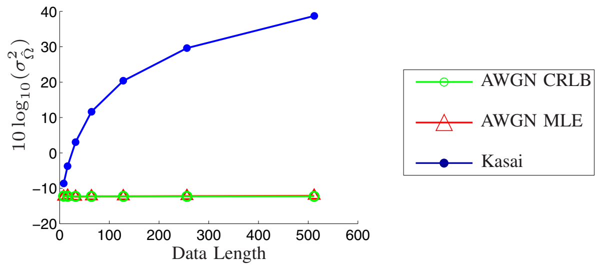

Fig. 3: MSE of Kasai and AWGN ML estimators with shot noise SNR scaling for a constant acquisition time of $T = 0.001s$ against data length. The SNR is 5 dB for $N = 32$. This noise scaling corresponds to maintaining a constant detected photon rate (power) as the acquisition rate is increased. For shot noise limited statistics, the SNR scales as T/N, hence the CRLB is constant with data length (7). The AWGN MLE performance closely matches that of the CRLB. The Kasai estimator performance deteriorates with increasing sampling rate.

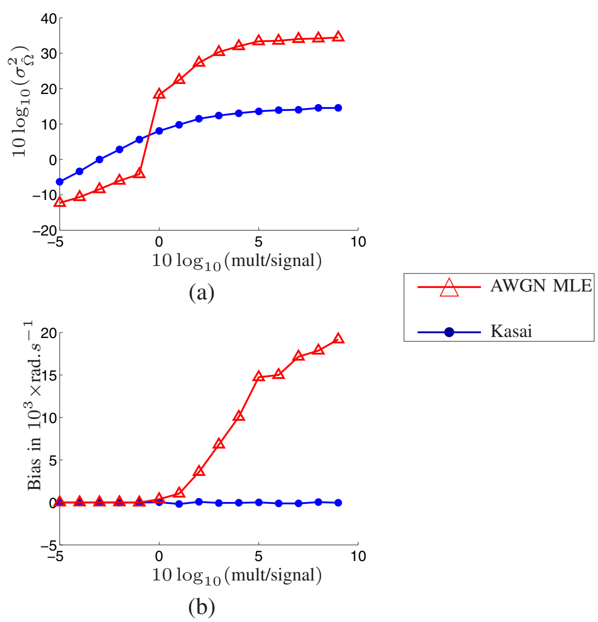

Fig. 4: (a) The sample variance of estimates for varying degrees of multiplicative noise with no additive noise for a data length of $N = 32$, and an acquisition time of $T = 0.001s$. The x-axis shows the ratio of multiplicative noise components to signal components in decibels. At roughly -0.5 dB the estimators have equal performance. From this we can see that the Kasai estimator is more robust against multiplicative decorrelation noise than the AWGN MLE.
(b) Estimator bias in 10^3 radians per second with SNR.

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REFERENCES

[1] C. Kasai, K. Namekawa, A. Koyano, and R. Omoto, "Real-time two-dimensional blood flow imaging using an autocorrelation technique," *IEEE Trans. Sonics Ultrason.*, vol. 32, no. 3, pp. 458–464, May 1985.

[2] A. C. Chan, E. Y. Lam, and V. J. Srinivasan, "Comparison of kasai autocorrelation and maximum likelihood estimators for doppler optical coherence tomography," *IEEE Trans. Med. Imag., (in submission)*, 2012.

[3] R. Aaslid, T.-M. Markwalder, and H. Nornes, "Noninvasive transcranial doppler ultrasound recording of flow velocity in basal cerebral arteries," *J. Neurosurg.*, vol. 57, no. 6, pp. 769–774, 1982.

[4] O. Rossvoll and L. K. Hatle, "Pulmonary venous flow velocities recorded by transthoracic doppler ultrasound: Relation to left ventricular diastolic pressures," *J. Am. Coll. Cardiol.*, vol. 21, no. 7, pp. 1687–1696, 1993.

[5] D. H. Evans and W. McDicken, *Doppler Ultrasound: Physics, Instrumentation and Signal Processing*, 2nd ed. Chichester: Wiley, 2000.

[6] J. A. Izatt, M. D. Kulkarni, S. Yazdanfar, J. K. Barton, and A. J. Welch, "In vivo bidirectional color doppler flow imaging of picoliter blood volumes using optical coherence tomography," *Opt. Lett.*, vol. 22, no. 18, pp. 1439–1441, September 1997.

[7] Z. Chen, T. E. Milner, D. Dave, and J. S. Nelson, "Optical doppler tomographic imaging of fluid flow velocity in highly scattering media," *Opt. Lett.*, vol. 22, no. 1, pp. 64–66, January 1997.

[8] T. Schmoll, C. Kolbitsch, and R. A. Leitgeb, "Ultra-high-speed volumetric tomography of human retinal blood flow," *Opt. Express*, vol. 17, no. 5, pp. 4166–4176, March 2009.

[9] E. L. Lehmann and G. Casella, *Theory of Point Estimation*, 2nd ed. Springer-Verlag, 1998.

[10] F. R. Hampel, *Robust Statistics: The Approach Based on Influence Functions*, 18th ed. New York: Wiley, 1986.

[11] P. J. Huber, *Robust Statistical Procedures*, 2nd ed. Society for Industrial and Applied Mathematics, 1996.

[12] B. Vakoc, G. Tearney, and B. Bouma, "Statistical properties of phase-decorrelation in phase-resolved doppler optical coherence tomography," *IEEE Trans. Med. Imag.*, vol. 28, no. 6, pp. 814–821, June 2009.